Exercise 55

If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

Solution

Use the definition of the derivative to find f'.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[2(x+h)^2 - (x+h)^3] - [2x^2 - x^3]}{h} \\
&= \lim_{h\to 0} \frac{[2(x^2 + 2xh + h^2) - (x^3 + 3x^2h + 3xh^2 + h^3)] - 2x^2 + x^3}{h} \\
&= \lim_{h\to 0} \frac{(2x^2 + 4xh + 2h^2 - x^3 - 3x^2h - 3xh^2 - h^3) - 2x^2 + x^3}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 3x^2h - 3xh^2 - h^3}{h} \\
&= \lim_{h\to 0} (4x + 2h - 3x^2 - 3xh - h^2) \\
&= 4x - 3x^2
\end{aligned}$$

Use the definition of the derivative again to find f''.

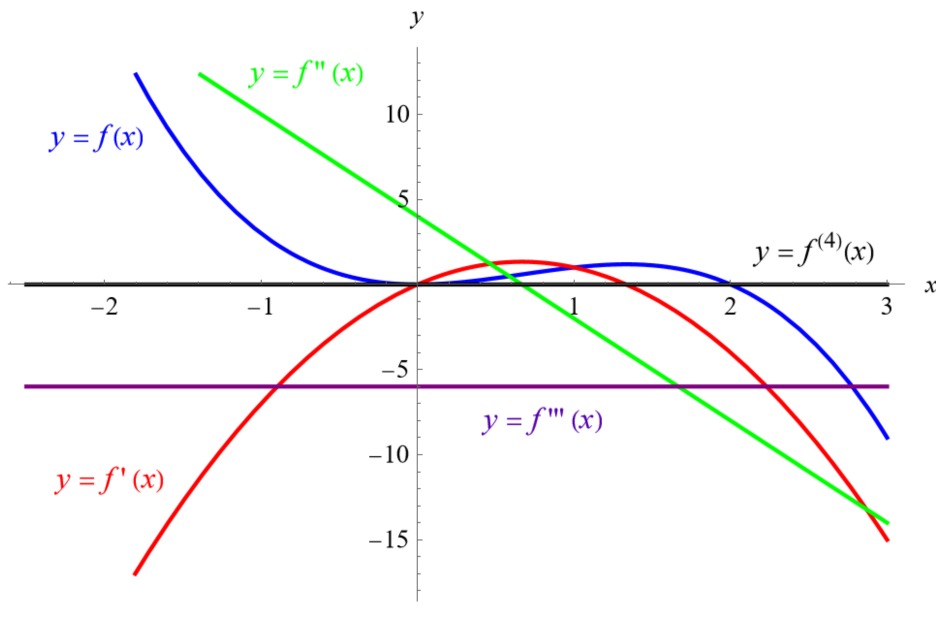
$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{[4(x+h) - 3(x+h)^2] - [4x - 3x^2]}{h} \\
&= \lim_{h\to 0} \frac{[4x + 4h - 3(x^2 + 2xh + h^2)] - 4x + 3x^2}{h} \\
&= \lim_{h\to 0} \frac{(4x + 4h - 3x^2 - 6xh - 3h^2) - 4x + 3x^2}{h} \\
&= \lim_{h\to 0} \frac{4h - 6xh - 3h^2}{h} \\
&= \lim_{h\to 0} (4 - 6x - 3h) \\
&= 4 - 6x
\end{aligned}$$

Use the definition of the derivative again to find f'''.

$$\begin{aligned} f'''(x) &= \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} \\ &= \lim_{h\to 0} \frac{[4 - 6(x+h)] - [4 - 6x]}{h} \\ &= \lim_{h\to 0} \frac{(4 - 6x - 6h) - 4 + 6x}{h} \\ &= \lim_{h\to 0} \frac{-6h}{h} \\ &= \lim_{h\to 0} (-6) \\ &= -6 \end{aligned}$$

Use the definition of the derivative again to find $f^{(4)}$.

$$\begin{aligned} f^{(4)}(x) &= \lim_{h\to 0} \frac{f'''(x+h) - f'''(x)}{h} \\ &= \lim_{h\to 0} \frac{(-6) - (-6)}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$



Notice that the derivative of a function is negative where the function is decreasing and positive where the function is increasing.